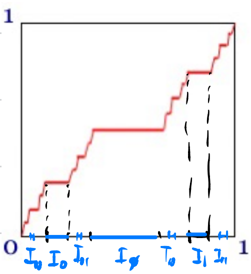
Metric Spaces and Topology
Lecture 9

Examples (continued). ○ Cantor function (devil's staircase)



Let $(I_s)_{s\in 2^{<\mathbb{N}}}$ be the sequence of open intervals, indexed by finite binary sequences, whose removal results in the Cantor set.

The Cantor function is a continuous function $f:[0,1]\to[0,1]$ s.t. $f|_{I_s} = 0.\underbrace{s_0 s_1 \dots s_n}_{\text{binary rep.}}1$.

$\boxed{\text{HW}}$

It follows abstractly (because of uniform continuity) that $\exists$ a unique such continuous function, but we'll give an explicit definition.

Fix $x \in [0,1]$ and we define $f(x)$ as follows. Take the ternary rep. of $x = 0.x_0x_1x_2x_3\dots$, $x_n \in \{0,1,2\}$. We take those representations favouring 1's, i.e.

yes $0.{*}{*}{*}1\,2222\dots = 0.{*}{*}{*}2\,000\dots$ no

no $0.{*}{*}{*}0\,222\dots = 0.{*}{*}{*}1\,000\dots$ yes

Then delete all the indices after the first 1 (but leave that 1). This results in a finite or infinite sequence. Replace all 2's with

1's in that sequence. The resulting sequence is binary, and $f(x)$ is the number with that binary representation.

E.g. $x = 0.0200201210 1102\ldots$

$$\Downarrow$$

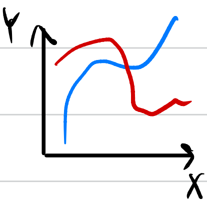
$$0.0200201 \rightsquigarrow 0.0100101 = f(x).$$

$$x = 0.02002020222 0 \ldots$$
$$f(x) = 0.01001010111 0\ldots$$

One can check that f is continuous (intuitively, this is because finitely many digits of $f(x)$ are determined by finitely many digits of x. $\boxed{\text{HW}}$

Spaces of Functions. Let (X, d_X), (Y, d_Y) be metric spaces, let Y^X denote the set of all functions $X \to Y$.

We'd like to define a metric on Y^X and here is an attempt:



$$d_u(f,g) := \sup_{x\in X} d_Y(f(x), g(x)),$$ call this the *uniform metric*.

The only issue is that d_u may take value ∞, so we call it an *extended metric*. $\boxed{\text{HW}}$ This indeed satisfies all metric axioms. Define an eq. rel. $\sim_{d_u}$ on Y^X by $f \sim_{d_u} g :\Leftrightarrow d_u(f,g) < \infty$.

Obs. Each $\sim_{d_u}$ eq. class is clopen. In particular, any union of eq. classes is again clopen (because the complement is also a union of eq. classes).

Examples.

- Let $B(X,Y)$ denote the set of all bounded functions $X \to Y$, where $f: X \to Y$ is called **bounded** if $\operatorname{diam} f(X) < \infty$.
  $B(X,Y)$ is one $\sim_{d_u}$ eq. class.
- $\boxed{\text{HW}}$ Let $C(X,Y)$ denote the set of cont. func. $X \to Y$. Show that $C(X,Y)$ is closed.
- The set $BC(X,Y)$ of bounded continuous function is closed and d_u is a metric on it.
  $$BC(X,Y) = \underbrace{B(X,Y)}_{\text{clopen}} \cap \underbrace{C(X,Y)}_{\text{closed}} \Rightarrow \text{closed}.$$

Theorem. Let X be a set and (Y,d) a metric space. If (Y,d) is complete, then so is (Y^X, d_u).

Proof. Let (f_n) be a d_u-Cauchy sequence in Y^X. Define $f: X \to Y$ by $f(x) := \lim_n f_n(x)$, which exists because $(f_n(x)) \subseteq Y$ is Cauchy in Y.

We show that $f_n \to f$ in the d_u-metric.

Fix $\varepsilon > 0$.

$$d_u(f_n, f) = \sup_{x \in X} d(f_n(x), f(x)).$$

It's enough to show that $\underbrace{\forall^\infty_n}_{\exists N\, \forall n \geq N} \underbrace{\forall x \; d(f_n(x), f(x)) < \varepsilon}_{\Downarrow \; d_u(f_n, f) \leq \varepsilon}$.

Let N be large enough so that $\forall n \geq N$ $d_u(f_{n+m}, f_n) < \frac{\varepsilon}{2}$.

Now fix $x \in X$. we know that $\forall^\infty_m$, this is $< \varepsilon/2$ but N may not work for this x.

$$d(f_n(x), f(x)) \leq d(f_n(x), f_{n+m}(x)) + d(f_{n+m}(x), f(x))$$

$\forall^\infty_m$ (depending on x) $< \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\square$

Corollary. If Y is a complete metric space, then

(a) $B(X,Y)$ and $BC(X,Y)$ complete metric spaces.

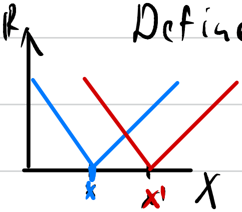
(b) $C(X,Y)$ is a complete extended metric space.

Alternative proof existence of completion (Kaplansky).

Every metric space (X,d) admits a completion.

Proof. We isometrically embed (X,d) into $\mathbb{R}^X$ and take the closure of the image of X in $\mathbb{R}^X$.

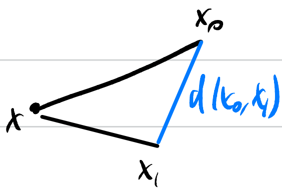
($\mathbb{R}^X$ is complete hence $\mathbb{R}$ is.)



Define $\iota : X \to \mathbb{R}^X$

$$x \mapsto f_x, \text{ where } f_x(y) := d(x,y).$$

Claim. ι is an isometry.



Proof. $$d_u(f_{x_0}, f_{x_1}) = \sup_{x\in X} |f_{x_0}(x) - f_{x_1}(x)|$$
$$= \sup_{x\in X} |d(x_0,x) - d(x_1,x)| \leq d(x_0,x_1)$$
and this is achieved by $x := x_0, x_1$. $\square$

Thus, we let $\hat{X} := \overline{\iota(X)}$ closure inside $\mathbb{R}^X$.

Because ι is an isometry and d is finitely valued metric, d_u on $\iota(X)$ is also a finitely valued metric, in other words, $\iota(X)$ is contained in a single $\sim_{d_u}$ eq. class, which hence is clopen, so $\overline{\iota(X)}$ is still in that one eq. class, hence d_u on $\overline{\iota(X)}$ is finite. $\square$